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Three-dimensional black holes from deformed anti de Sitter*

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Abstract: We present new exact three-dimensional black-string backgrounds, which contain both NS-NS and electromagnetic fields, and generalize the BTZ black holes and the black string studied by Horne and Horowitz. They are obtained as deformations of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. Black holes resulting from purely continuous deformations possess true curvature singularities. When discrete identifications are introduced, extra chronological singularities appear, which under certain circumstances turn out to be naked. The backgrounds at hand appear in the moduli space of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. Hence, they provide exact string backgrounds and allow for a more algebraical (CFT) description. This makes possible the determination of the spectrum of primaries.

Keywords: Conformal Field Models in String Theory Black Holes in String Theory.

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1. Introduction

The search for exact string backgrounds has been pursued over the past years from various perspectives. Those investigations are motivated by phenomenology, background-geometry analysis or, more recently, for understanding holography beyond the usual supergravity approximation.

Anti-de Sitter backgrounds have played an important role in many respects. Together with the spheres, they are the only maximally symmetric spaces appearing naturally in string theory. They arise as near-horizon geometries of distinguished brane configurations and offer the appropriate set up for studying little-string theory, black-hole physics, ...

The realization of anti-de Sitter spaces or spheres as string backgrounds requires non-vanishing fluxes, which account for the cosmological constant term in the low-energy equations of motion. In general, those fluxes are of the Ramond-Ramond type, hence no two-dimensional sigma-model is available. This happens indeed for $\mathrm{AdS}_5 \times S^5$ in type IIB or $\mathrm{AdS}_4 \times S^7$ in M-theory. For $\mathrm{AdS}_3 \times S^3 \times T^4$ (type IIA, B or heterotic), however, we have the option to switch on a Neveu-Schwarz antisymmetric tensor only. In this framework, the $\mathrm{AdS}_3 \times S^3$ is the target space of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ Wess-Zumino-Witten model. The latter has been studied extensively [1−6].

Three-dimensional anti-de Sitter space provides a good laboratory for studying many aspects of gravity and strings, including black-hole physics. Locally anti-de Sitter three-dimensional black holes are obtained by performing identifications in the original AdS_3 under discrete isometry subgroups [7−10]. Those black holes (BTZ) have mass and angular momentum. Generically, two horizons (inner and outer) mask the singularity, which turns out to be a chronological singularity rather than a genuine curvature singularity.

The two-dimensional sigma-model description of the AdS_3 plus Kalb-Ramond field background allows for exact conformal deformations, driven by integrable marginal operators [11−14, 6, 15−17]. In general, a subgroup of the original isometry group survives along those lines. Identification under discrete isometries is thus legitimate and provides a tool for investigating new and potentially interesting "deformed BTZ" geometries. The latter may or may not be viable black holes, whereas black holes may also appear by just deforming AdS_3 without further surgery [18].

The aim of the present work is to clarify those issues, and reach a global point of view on the geometries that emerge from the $\mathrm{SL}(2,\mathbb{R})$ WZW model, by using the above techniques. This will allow us to introduce new three-dimensional black hole backgrounds that in general involve the presence of an electric field. For these theories we give a complete CFT description, including an explicit expression for the spectrum of string primaries. In particular, the usual black string background [18] will appear in this terms as a special vanishing-field limit. Carrying on identifications *à la* BTZ on these geometries will let us obtain more black string and/or black hole backgrounds, generalizing the one in [7] and in [18], for which we again provide a CFT description. Not all the backgrounds could be adapted to support the discrete identifications. This will be stated in terms of a consistency condition that has to be satisfied in order to avoid the presence of naked (causal) singularities.

We will start with a quick overview of various distinct methods based on Wess-Zumino-Witten models and aiming at generating new exact CFTs, that turn out to be equivalent to each other. We will in particular exhibit their effect on the $\mathrm{SL}(2,\mathbb{R})$ WZW model. These results enable us to recast in section 3 the three-dimensional black-string solution of [18], as a patchwork of marginal deformations of the $\mathrm{SL}(2,\mathbb{R})$ WZW model. We clarify in this way the role of the mass and charge parameters of the black string.

Section 4 is devoted to a two-parameter deformation of $SL(2,\mathbb{R})$. This leads to a new family of black strings, with NS-NS and electric field. We study the causal structure of these black holes as well as their various charges. They exhibit genuine curvature singularity hidden behind horizons. In section 5 we proceed with discrete identifications as a solution-generating procedure applied to the deformed AdS_3 — wherever it is allowed by residual symmetries.

After having stated the consistency conditions to be fulfilled in order to avoid naked singularities, we find that time-like chronological singularities protected by two horizons are possible, while light-like singularities with only one horizon appear as a limiting case. Finally, in section 6 we determine the spectrum of primaries, using standard CFT techniques.

2. Deformed WZW models: various perspectives

The power of WZW models resides in the symmetries of the theory. Those impose strong constraints which allow quantum integrability as well as a faithful description in terms of space-time fields whose renormalization properties (at every order in α') are easily kept under control [19–21].

It is hence interesting to study the moduli space for these models, aiming at finding less symmetric (and more interesting) structures, that will hopefully enjoy analogous integrability and space-time properties.

2.1 Algebraic structure of current-current deformations

In this spirit one can consider marginal deformations of the WZW models obtained in terms of $(1,1)$ operators built as bilinears in the currents:

$$\mathcal{O}(z,\bar{z}) = \sum_{ij} c_{ij} J^i(z)\,\tilde{J}^j(\bar{z}), \tag{2.1}$$

where $J^i(z)$ and $\tilde{J}^j(\bar{z})$ are respectively left- and right-moving currents. It is known that this operator represents a truly marginal deformation if the parameter matrix c_{ij} satisfies appropriate constraints [11], which are automatically satisfied for any value of c_{ij}, whatever the algebra, if J^i and $\tilde{J}^j$ live on a torus. Hence, we get as moduli space continuous surfaces of exact models.[1]

From the CFT point of view, it is known [14] that the effect of the deformation is completely captured by an $O(d,\bar{d})$ pseudo-orthogonal transformation of the charge lattice $\Lambda \subset \mathfrak{h}^* \times \bar{\mathfrak{h}}^*$ of the abelian sector of the theory ($\mathfrak{h} \subset \mathfrak{g}$ and $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$ being abelian subalgebras of the undeformed WZW model $\mathfrak{g}\times\bar{\mathfrak{g}}$ algebra). Moreover, since the charges only characterize the $\mathfrak{h}\times\bar{\mathfrak{h}}$ modules up to automorphisms of the algebras, $O(d)\times O(\bar{d})$ transformations don't

[1] Although for special values of the level k the theory contains other operators with the right conformal weights, it is believed that only current-current operators give rise to truly marginal deformations, i.e. operators that remain marginal for finite values of the deformation parameter.

change the CFT. Hence the deformation space is given by:

$$D_{\mathfrak{h},\bar{\mathfrak{h}}} \sim O(d,\bar{d}) / \left(O(d) \times O(\bar{d})\right). \tag{2.2}$$

The moduli space is obtained out of $D_{\mathfrak{h},\bar{\mathfrak{h}}}$ after the identification of the points giving equivalent CFTs.[2]

In the case of WZW models on compact groups, all maximal abelian subgroups are pairwise conjugated by inner automorphisms. This implies that the complete deformation space is $D = O(d,d)/\left(O(d) \times O(d)\right)$ where d is the rank of the group. The story is different for non-semisimple algebras, whose moduli space is larger, since we get different $O(d,\bar{d})/\left(O(d) \times O(\bar{d})\right)$ deformation spaces for each (inequivalent) choice of the abelian subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$.

An alternative way of describing current-current deformations comes from the so-called parafermion decomposition. The highest-weight representation for a $\hat{\mathfrak{g}}_k$ graded algebra can be decomposed into highest-weight modules of a Cartan subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}_k$ as follows [22, 23]:

$$\mathcal{V}_{\hat{\lambda}} \simeq \bigoplus_{\mu \in \Gamma_k} \mathcal{V}_{\hat{\lambda},\mu} \otimes \bigoplus_{\delta \in Q_l(\mathfrak{g})} \mathcal{V}_{\mu + k\delta}\,, \tag{2.3}$$

where $\hat{\lambda}$ is an integrable weight of $\hat{\mathfrak{g}}_k$, $\mathcal{V}_{\hat{\lambda},\mu}$ is the highest-weight module for the generalized $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}$ parafermion, $Q_l(\mathfrak{g})$ is the long-root lattice and $\Gamma_k = P(\mathfrak{g})/Q_l(\mathfrak{g})$ with $P(\mathfrak{g})$ the weight lattice. As a consequence, the WZW model based on $\hat{\mathfrak{g}}_k$ can be represented as an orbifold model:

$$\hat{\mathfrak{g}}_k \simeq \left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}} \otimes t_{\Lambda_k}\right)/\Gamma_k\,, \tag{2.4}$$

where t_{Λ_k} is a toroidal CFT with charge lattice, included in the $\hat{\mathfrak{g}}_k$ one, defined as $\Lambda_k = \{(\mu,\bar{\mu}) \in P(\hat{\mathfrak{g}}) \times P(\hat{\mathfrak{g}}) | \mu - \bar{\mu} = kQ_l(\hat{\mathfrak{g}})\}$. The advantage given by using this representation relies on the fact that Γ_k acts trivially on the coset and toroidal model algebras; then, if we identify $\hat{\mathfrak{h}}$ and $\hat{\bar{\mathfrak{h}}}$ with the graded algebras of t_{Λ_k}, the deformation only acts on the toroidal lattice and the deformed model can again be represented as an orbifold:

$$\hat{\mathfrak{g}}_k(\mathcal{O}) \simeq \left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}} \otimes t_{\mathcal{O}\Lambda_k}\right)/\Gamma_k\,, \tag{2.5}$$

where $\mathcal{O}$ is an operator in the moduli space. This representation is specially useful because it allows to easily single out the sector of the theory that is affected by the deformation. As we'll see in the next section this simplifies the task of writing the corresponding lagrangian.

In the following we will separate (somehow arbitrarily) this kind of deformations into two categories: those who give rise to *symmetric* deformations, i.e. the ones where $c_{ij} = \delta_{ij}$ and $J^i(z)$ and $\tilde{J}^j(\bar{z})$ represent the same current in the two chiral sectors of the theory and the *asymmetric* ones where the currents are different and in general correspond to different subalgebras. In some ways this distinction is arbitrary, since both symmetric and asymmetric deformations act as $O\left(d,\bar{d}\right)$ rotations on the background fields. It is nonetheless interesting to single out the asymmetric case. In the particular situation,

[2]Although we will concentrate on WZW models it is worth to emphasize that this construction is more general.

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when one of the two currents belongs to an internal U(1) (coming from the gauge sector in the heterotic or simply from any U(1) subalgebra in the type II), it is particularly simple to study the effect of the deformation, even from the space-time field point of view; there, the expressions for the background fields are exact (at all order in α' and for every value of the level k) [16].

2.2 Background fields and symmetric deformations

General construction. Symmetric deformations (also called *gravitational*) are those that have received by far the most attention in literature. Specializing eq. (2.1) to the case of one only current we can write the small deformation lagrangian as:

$$S = S_{\mathrm{WZW}} + \delta\kappa^2 \int \mathrm{d}^2 z \, J(z)\bar{J}(\bar{z}) \,. \tag{2.6}$$

This infinitesimal deformation has to be integrated in order to give a lagrangian interpretation to the CFT described in the previous section. Different approaches are possible, exploiting the different possible representations described above.

- A possible way consists in implementing an $O(d,d)$ rotation on the background fields [12]. More precisely, one has to identify a coordinate system in which the background fields are independent of d space dimensions and metric and B field are written in a block diagonal form. In this way the following matrix is defined:

  $$M = \left(\begin{array}{c|c} \hat{g}^{-1} & -\hat{g}^{-1}\hat{B} \\ \hline \hat{B}\hat{g}^{-1} & \hat{g} - \hat{B}\hat{g}^{-1}\hat{B} \end{array} \right), \tag{2.7}$$

  where $\hat{g}$ and $\hat{B}$ are the pull-backs of the metric and Kalb-Ramond field on the p selected directions. Then the action of the $O(d,d)$ group on these fields and dilaton is given by:

  $$M \to M' = \Omega M \Omega^t \,, \tag{2.8}$$

  $$\Phi \to \Phi' = \Phi - \frac{1}{4} \log \left(\frac{\det \hat{g}}{\det \hat{g}'} \right), \tag{2.9}$$

  where $\hat{g}'$ is the metric after the transformation (2.8) and $\Omega \in O(d,d)$. It must be emphasized that this transformation rules are valid at the lowest order in α' (but at all orders in the deformation parameters). So, although the model is exact, as we learn from the CFT side, the field expressions that we find only are true at leading order in α'.

- An alternative approach uses the parafermion representation eq. (2.5) (see e.g. [14]). In practice this amounts to writing an action as the sum of the G/H parafermion and a deformed H part and finding the appropriate T-duality transformation (realizing the orbifold) such that for zero deformation the WZW on G is recovered, in accordance with eq. (2.4).

- Finally, another point of view (inspired by the parafermionic representation), consists in identifying the deformed model with a $(G \times H)/H$ coset model, in which the embedding of the dividing group has a component in both factors [13]. The gauging of the component in G gives the parafermionic sector, the gauging of the component in H gives the deformed toroidal sector and the coupling term (originating from the quadratic structure in the fields introduced for the gauging) corresponds to the orbifold projection.[3]

The $\mathrm{SL}(2,\mathbb{R})$ case. In the present work, we want to concentrate on the deformations of $\mathrm{SL}(2,\mathbb{R})$. Symmetric deformations of this WZW model are known in the literature. The group manifold of $\mathrm{SL}(2,\mathbb{R})$ is anti de Sitter in three dimensions. Metric and antisymmetric tensor read (in Euler coordinates, see appendix A.2):

$$\mathrm{d}s^2 = L^2\left[\mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\phi^2 - \cosh^2\rho\,\mathrm{d}\tau^2\right], \tag{2.10a}$$

$$H_{[3]} = L^2 \sinh 2\rho\,\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\tau\,, \tag{2.10b}$$

with L related to the level of $\mathrm{SL}(2,\mathbb{R})_k$ as usual: $L = \sqrt{k+2}$. In the case at hand, three different lines of symmetric deformations arise due to the presence of time-like (J^3, $\bar{J}^3$), space-like (J^1, $\bar{J}^1$, J^2, $\bar{J}^2$), or null generators [14, 25, 6]. The residual isometry is $\mathrm{U}(1)\times\mathrm{U}(1)$ that can be time-like (L_3, R_3), space-like (L_2, R_2) or null (L_1+L_3, R_1+R_3) depending on the deformation under consideration.

The *elliptic deformation* is driven by the $J^3\bar{J}^3$ bilinear. At first order in α' the background fields are given by:[4]

$$\mathrm{d}s^2 = k\left[\mathrm{d}\rho^2 + \frac{\sinh^2\rho\,\mathrm{d}\phi^2 - \kappa_3^2\cosh^2\rho\,\mathrm{d}\tau^2}{\Theta_{\kappa_3}(\rho)}\right], \tag{2.11a}$$

$$H_{[3]} = k\frac{\kappa_3^2\sinh 2\rho}{\Theta_{\kappa_3}(\rho)^2}\,\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\tau\,, \tag{2.11b}$$

$$\mathrm{e}^{-2\Phi} = \frac{\Theta_{\kappa_3}(\rho)}{\kappa_3}. \tag{2.11c}$$

where $\Theta_{\kappa_3}(\rho) = \cosh^2\rho - \kappa_3^2\sinh^2\rho$ and, of course, Φ is defined up to an additive constant. At extreme deformation ($\kappa_3^2 \to 0$), a time-like direction decouples and we are left with the axial[5] $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)_{\mathrm{time}}$. The target space of the latter is the *cigar* geometry (also called euclidean two-dimensional black hole):

$$\mathrm{e}^{-2\Phi} \sim \cosh^2\rho\,, \tag{2.12}$$

$$\mathrm{d}s^2 = k\left[\mathrm{d}\rho^2 + \tanh^2\rho\,\mathrm{d}\phi^2\right], \tag{2.13}$$

$(0 \le \rho < \infty$ and $0 \le \phi \le 2\pi)$.

[3] An instanton-correction-aware technique that should overcome the first order in α' limitation for gauged models has been proposed in [24]. In principle this can be used to get an all-order exact background when we write the deformation as a gauged model. We will not expand further in this direction, that could nevertheless be useful to address issues such as the stability of the black string (see section 3).

[4] The extra index "3" in the deformation parameter κ reminds that the deformation refers here to $J^3\bar{J}^3$.

[5] The deformation parameter has two T-dual branches. The extreme values of deformation correspond to the axial or vector gaugings. The vector gauging leads to the *trumpet*. For the $\mathrm{SU}(2)_k/\mathrm{U}(1)$, both gaugings correspond to the *bell*.

Similarly, with $J^2 \bar{J}^2$ one generates the *hyperbolic deformation*. This allows to reach the lorentzian two-dimensional black hole times a free space-like line. Using the coordinates defined in eq. (A.8), we find:

$$\mathrm{d}s^2 = k\left[-\,\mathrm{d}\beta^2 + \frac{\sin^2\beta\,\mathrm{d}\varphi^2 + \kappa_2^2\cos^2\beta\,\mathrm{d}\psi^2}{\Delta_{\kappa_2}(\beta)}\right], \tag{2.14a}$$

$$H_{[3]} = k\frac{\kappa_2^2\sin 2\beta}{\Delta_{\kappa_2}(\beta)^2}\,\mathrm{d}\beta\wedge\mathrm{d}\psi\wedge\mathrm{d}\phi\,, \tag{2.14b}$$

$$\mathrm{e}^{-2\Phi} = \frac{\Delta_{\kappa_2}(\beta)}{\kappa_2}\,, \tag{2.14c}$$

where $\Delta_{\kappa_2}(\beta) = \cos^2\beta + \kappa_2^2\sin^2\beta$. This coordinate patch does not cover the full AdS_3. We will expand on this line in section 3.

Finally, the bilinear $\left(J^1 + J^3\right)\left(\bar{J}^1 + \bar{J}^3\right)$ generates the *parabolic deformation*. Using Poincaré coordinates (eqs. (A.11)–(A.13))[6] we obtain:

$$\mathrm{d}s^2 = k\left[\frac{\mathrm{d}u^2}{u^2} + \frac{\mathrm{d}X^2 - \,\mathrm{d}T^2}{u^2 + 1/\nu}\right], \tag{2.15a}$$

$$H_{[3]} = k\frac{2u}{\left(u^2+1/\nu\right)^2}\,\mathrm{d}u\wedge\mathrm{d}T\wedge\mathrm{d}X\,, \tag{2.15b}$$

$$\mathrm{e}^{-2\Phi} = \frac{u^2+1/\nu}{u^2}\,. \tag{2.15c}$$

The deformation parameter is $1/\nu$. At infinite value of the parameter ν, we recover pure AdS_3; for $\nu \to 0$, a whole light-cone decouples and we are left with a single direction and a dilaton field, linear in this direction.

The physical interpretation of the parabolic deformation is far reaching, when AdS_3 is considered in the framework of the NS5/F1 near-horizon background, $\mathrm{AdS}_3 \times S^3 \times T^4$. In this physical set-up, the parameter ν is the density of F1's (number of fundamental strings over the volume of the four-torus T^4) [6, 26]. At infinite density, the background is indeed $\mathrm{AdS}_3 \times S^3 \times T^4$. At null density, the geometry becomes $\mathbb{R}^{1,2} \times S^3 \times T^4$ plus a linear dilaton and a three-form on the S^3.

2.3 Background fields and asymmetric deformations

General construction. Consider the case of $G = G' \times \mathrm{U}(1)^r$, $H = \mathrm{U}(1)^r$ where $r = \mathrm{rank}\,(G)$ embedded such as $\epsilon_{\mathrm{L}}\,(H) \subset G'$ and $\epsilon_{\mathrm{R}}\,(H) = G'' = \mathrm{U}(1)^r$. To clarify the notation we can write the deformation operator as:

$$\mathcal{O}(z,\bar{z}) = \sum_{a=1}^{r} \mathrm{H}_a J^a\,(z)\;\bar{\partial}X^a \tag{2.16}$$

where $X^a\,(z,\bar{z})$ results from the bosonisation of the right current. Using e.g. Kaluza-Klein reduction [27, 28, 17], one shows that the effect of the deformation on the background fields

[6] Note that $x^{\pm} = X \pm T$.

(identified as those living in the G' sector) is the following:

$$G_{\mu\nu} = \mathring{G}_{\mu\nu} - 2 \sum_{i=a}^{r} \mathrm{H}_a^2 J_\mu^a J_\nu^a \,, \tag{2.17a}$$

$$B_{\mu\nu} = \mathring{B}_{\mu\nu} \,, \tag{2.17b}$$

$$A_\mu^a = \mathrm{H}_a \sqrt{\frac{2k}{k_g}} J_\mu^a \,, \tag{2.17c}$$

where $\mathring{G}_{\mu\nu}$ and $\mathring{B}_{\mu\nu}$ are the initial, unperturbed background fields that are expressed in terms of the $g \in G'$ group element as follows:

$$\mathring{G}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = \left\langle g^{-1} \, \mathrm{d}g, g^{-1} \, \mathrm{d}g \right\rangle \,, \tag{2.18a}$$

$$\mathring{B}_{\mu\nu} \, \mathrm{d}x^\mu \wedge x^\nu = g^{-1} \, \mathrm{d}g \wedge g^{-1} \, \mathrm{d}g \,. \tag{2.18b}$$

No dilaton is present (as a consequence of the fact that the Ricci scalar for these deformed systems remains constant) and these semiclassical solutions can be promoted to exact ones just by remarking that the effect of the renormalisation simply boils down to the shift $k \to k + c_{G'}$ where $c_{G'}$ is the dual Coxeter number, just as in the case of the unperturbed WZW model.

The $\mathrm{SL}\,(2,\mathbb{R})$ **case.** We now apply the above to the $\mathrm{SL}(2,\mathbb{R})$ case. As previously, three asymmetric deformations are available: the elliptic, the hyperbolic and the parabolic.

The *elliptic deformation* is generated by a bilinear where the left current is an $\mathrm{SL}(2,\mathbb{R})_k$ time-like current. The background field is magnetic and the residual symmetry is $\mathrm{U}(1)_{\mathrm{time}} \times \mathrm{SL}(2,\mathbb{R})$ generated by $\{L_3, R_1, R_2, R_3\}$ (see appendix A.2). The metric reads (in elliptic coordinates):

$$\mathrm{d}s^2 = \frac{k}{4} \left[\mathrm{d}\rho^2 + \cosh^2 \rho \, \mathrm{d}\phi^2 - \left(1 + 2\mathrm{H}^2\right) \left(\mathrm{d}t + \sinh \rho \, \mathrm{d}\phi \right)^2 \right] , \tag{2.19}$$

where ∂_t is the Killing vector associated with the $\mathrm{U}(1)_{\mathrm{time}}$. This AdS_3 deformation was studied in [29] as a *squashed anti de Sitter* and in [15, 16] from the string theory point of view. It has curvature

$$\mathcal{R} = -\frac{2}{k} (3 - 2\mathrm{H}^2) \,. \tag{2.20}$$

Here, it comes as an *exact string solution* (provided $k \to k + 2$) together with an NS three-form and a magnetic field:

$$H_{[3]} = \mathrm{d}B - \frac{k_g}{4} A \wedge \mathrm{d}A = -\frac{k}{4} \left(1 + 2\mathrm{H}^2\right) \cosh \rho \, \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}t \,, \tag{2.21a}$$

$$A = \mathrm{H} \sqrt{\frac{2k}{k_g}} \left(\mathrm{d}t + \sinh \rho \, \mathrm{d}\phi \right) . \tag{2.21b}$$

For $\mathrm{H}^2 > 0$ (unitary region), the above metric is pathological because it has topologically trivial closed time-like curves passing through any point of the manifold. Actually, for $\mathrm{H}^2 = 1/2$ we recover exactly the Gödel space, which is a well-known example of pathological solution of Einstein-Maxwell equations.

The *hyperbolic deformation* can be studied in a similar fashion, where the left current in the bilinear is an $\mathrm{SL}(2,\mathbb{R})_k$ space-like current. In hyperbolic coordinates:

$$\mathrm{d}s^2 = \frac{k}{4}\left[\mathrm{d}r^2 - \cosh^2 r\,\mathrm{d}\tau^2 + \left(1-2\textsc{h}^2\right)\left(\mathrm{d}x + \sinh r\,\mathrm{d}\tau\right)^2\right], \tag{2.22}$$

where ∂_x generates a $\mathrm{U}(1)_{\text{space}}$. The total residual symmetry is $\mathrm{U}(1)_{\text{space}} \times \mathrm{SL}(2,\mathbb{R})$, generated by $\{L_2, R_1, R_2, R_3\}$, and

$$\mathcal{R} = -\frac{2}{k}\left(3+2\textsc{h}^2\right). \tag{2.23}$$

The complete string background now has an NS three-form and an electric field:

$$H_{[3]} = \frac{k}{4}\left(1-2\textsc{h}^2\right)\cosh r\,\mathrm{d}r\wedge\mathrm{d}\tau\wedge\mathrm{d}x\,, \tag{2.24a}$$

$$A = \textsc{h}\sqrt{\frac{2k}{k_g}}\left(\mathrm{d}x + \sinh r\,\mathrm{d}\tau\right). \tag{2.24b}$$

The background at hand is free of closed time-like curves. The squashed AdS_3 is now obtained by going to the AdS_3 picture as an S^1 fibration over an AdS_2 base, and modifying the S^1 fiber. The magnitude of the electric field is limited at $\textsc{h}^2_{\text{max}} = 1/2$, where it causes the degeneration of the fiber, and we are left with an AdS_2 background with an electric monopole; in other words, a geometric coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)_{\text{space}}$.

The string spectrum of the above deformation is accessible by conformal-field-theory methods. It is free of tachyons and a whole tower of states decouples at the critical values of the electric fields. Details are available in [16].

Finally, the *parabolic deformation* is generated by a null $\mathrm{SL}(2,\mathbb{R})_k$ current times some internal right-moving current. The deformed metric reads, in Poincaré coordinates:

$$\mathrm{d}s^2 = k\left[\frac{\mathrm{d}u^2}{u^2} + \frac{\mathrm{d}x^+\,\mathrm{d}x^-}{u^2} - 2\textsc{h}^2\left(\frac{\mathrm{d}x^+}{u^2}\right)^2\right], \tag{2.25}$$

and the curvature remains unaltered $\mathcal{R} = -6/k$. This is not surprising since the resulting geometry is a plane-wave like deformation of AdS_3. The residual symmetry is $\mathrm{U}(1)_{\text{null}} \times \mathrm{SL}(2,\mathbb{R})$, where the $\mathrm{U}(1)_{\text{null}}$ is generated by $\partial_- = -L_1 - L_3$.

The parabolic deformation is somehow peculiar. Although it is continuous, the deformation parameter can always be re-absorbed by a redefinition of the coordinates:[7] $x^+ \to x^+/\left|\textsc{h}\right|$ and $x^- \to x^-\left|\textsc{h}\right|$. Put differently, there are only three truly different options: $\textsc{h}^2 = 0, 1$. No limiting geometry emerges in the case at hand.

As expected, the gravitational background is accompanied by an NS three-form (unaltered) and an electromagnetic wave:

$$A = 2\sqrt{\frac{2k}{k_g}}\,\textsc{h}\frac{\mathrm{d}x^+}{u^2}. \tag{2.26}$$

[7] This statement holds as long as these coordinates are not compact. After discrete identifications have been imposed (see section 5.1)), $\textsc{h}$ becomes a genuine continuous parameter.

A final remark is in order here, which holds for all three asymmetric deformations of $\mathrm{SL}(2,\mathbb{R})$. The background electric or magnetic fields that appear in these solutions (eqs. (2.21b), (2.24b) and (2.26)) diverge at the boundary of the corresponding spaces. Hence, these fields cannot be considered as originating from localized charges.

3. The three-dimensional black string revisited

The AdS_3 moduli space contains black hole geometries. This has been known since the most celebrated of them — the two-dimensional $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ black hole — was found by Witten [30, 31]. Generalisations of these constructions to higher dimensions have been considered in [18, 32 – 34]. The three-dimensional black string [18, 35, 36] has attracted much attention, for it provides an alternative to the Schwarzschild black hole in three-dimensional asymptotically flat geometries.[8] In this section we want to show how this black string can be interpreted in terms of marginal deformations of $\mathrm{SL}(2,\mathbb{R})$, which will enable us to give an expression for its string primary states (section 6).

In [18] the black string was obtained as an $(\mathrm{SL}(2,\mathbb{R})\times\mathbb{R})/\mathbb{R}$ gauged model. More precisely, expressing $g\in\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}$ as:

$$g=\begin{pmatrix} a & u & 0\\ -v & b & 0\\ 0 & 0 & \mathrm{e}^{x}\end{pmatrix}, \tag{3.1}$$

the left and right embeddings of the $\mathbb{R}$ subgroup are identical and given by:

$$\epsilon_{L/R}:\mathbb{R}\to\mathrm{SL}\left(2,\mathbb{R}\right)\times\mathbb{R} \tag{3.2}$$

$$\lambda\mapsto\begin{pmatrix}\mathrm{e}^{\frac{1}{\sqrt{\lambda^2+2}}} & 0 & 0\\ 0 & \mathrm{e}^{-\frac{1}{\sqrt{\lambda^2+2}}} & 0\\ 0 & 0 & \mathrm{e}^{\frac{\lambda}{\sqrt{\lambda^2+2}}}\end{pmatrix}. \tag{3.3}$$

From the discussion in section 2.2, we see that performing this gauging is just one of the possible ways to recover the $J^2\bar{J}^2$ symmetrically deformed $\mathrm{SL}\left(2,\mathbb{R}\right)$ geometry. More specifically, since the gauged symmetry is axial ($g\to hgh$), it corresponds (in our notation) to the $\kappa_2<1$ branch of the deformed geometry[9] in eq. (2.14a). One can find a coordinate transformation allowing to pass from the usual black-string solution

$$\mathrm{d}s^2=\frac{k}{4}\left[-\left(1-\frac{1}{r}\right)\mathrm{d}t^2+\left(1-\frac{\mu^2}{r}\right)\mathrm{d}x^2+\left(1-\frac{1}{r}\right)^{-1}\left(1-\frac{\mu^2}{r}\right)^{-1}\frac{\mathrm{d}r^2}{r^2}\right], \tag{3.4a}$$

$$H=\frac{k}{4}\frac{\mu}{r}\,\mathrm{d}r\wedge\mathrm{d}x\wedge\mathrm{d}t\,, \tag{3.4b}$$

$$\mathrm{e}^{2\Phi}=\frac{\mu}{r} \tag{3.4c}$$

[8] Remember that the *no hair* theorem doesn't hold in three dimensions [37 – 39].

[9] The $R\gtrless 1$ convention is not univocal in literature.

μ	name	$\mathrm{d}t^2$: $-\left(1-\frac{1}{r}\right)$	$\mathrm{d}x^2$: $1-\frac{\mu^2}{r}$	$\mathrm{d}r^2$: $\left(1-\frac{1}{r}\right)^{-1}\left(1-\frac{\mu^2}{r}\right)^{-1}$	range	CFT interpretation
$\mu^2 > 1$	(c^+)	$-$	$+$	$+$	$r > \mu^2$	$J^3\bar{J}^3$, $\kappa_3 > 1$
	(b^+)	$-$	$-$	$-$	$1 < r < \mu^2$	
	(a^+)	$+$	$-$	$+$	$0 < r < 1$	$J^3\bar{J}^3$, $\kappa_3 < 1$
$\mu^2 < 1$	(a^-)	$+$	$-$	$+$	$0 < r < \mu^2$	$J^2\bar{J}^2$, $\kappa_2 < 1$
	(b^-)	$+$	$+$	$-$	$\mu^2 < r < 1$	
	(c^-)	$-$	$+$	$+$	$r > 1$	

Table 1: Signature for the black-string metric as a function of r, for $\mu^2 \gtrless 1$.

to our (local) coordinate system, eq. (2.14). The attentive reader might now be puzzled by this equivalence between a one-parameter model such as the symmetrically deformed model and a two-parameter one such as the black string in its usual coordinates (in eqs. (3.4) we redefined the r coordinate as $r \to r/M$ and then set $\mu = Q/M$ with respect to the conventions in [18]). A point that it is interesting to make here is that although, out of physical considerations, the black string is usually described in terms of two parameters (mass and charge), the only physically distinguishable parameter is their ratio $\mu = Q/M$ that coincides with our κ_2 parameter. In section 4 we will introduce a different (double) deformation, this time giving rise to a black hole geometry depending on two actual parameters (one of which being related to an additional electric field).

As we remarked above, the axial gauging construction only applies for $\mu < 1$, while, in order to obtain the other $\kappa_2 > 1$ branch of the $J^2\bar{J}^2$ deformation, one should perform a vector gauging. On the other hand, this operation, that would be justified by a CFT point of view, is not natural when one takes a more geometrical point of view and writes the black string metric as in eq. (3.4a). In the latter, one can study the signature of the metric as a function of r in the two regions $\mu^2 \gtrless 1$, and find the physically sensible regions (see table 1).

Our observations are the following:

- The $\mu^2 < 1$ branch always has the correct $(-,+,+)$ signature for any value of r, with the two special values $r = 1$ and $r = \mu^2$ marking the presence of the horizons that hide the singularity in $r = 0$.

- The $\mu^2 > 1$ branch is different. In particular we see that there are two regions: (a^+) for $0 < r < 1$ and (c^+) for $r > \mu^2$ where the signature is that of a physical space.

A fact deserves to be emphasized here: one should notice that while for $\mu^2 < 1$ we obtain three different regions of the same space, for $\mu^2 > 1$ what we show in table 1 really are three different spaces and the proposed ranges for r are just an effect of the chosen parameterization. The (a^+), $\kappa_3 < 1$ and (c^+), $\kappa_3 > 1$ branches are different spaces and not different regions of the same one and one can choose in which one to go when continuing to $\mu > 1$.

But there is more. The $\mu^2 > 1$ region is obtained via an analytic continuation with respect to the other branch, and this analytic continuation is precisely the one that interchanges the roles of the J^2 and the J^3 currents. As a result, we pass from the $J^2\bar{J}^2$ line to the $J^3\bar{J}^3$ line. More precisely the (c^+) region describes the "singular" $\kappa_3 > 1$ branch of the $J^3\bar{J}^3$ deformation (i.e. the branch that includes the $r = 0$ singularity) and the (a^+) region describes the regular $\kappa_3 < 1$ branch that has the *cigar* geometry as $\kappa_3 \to 0$ limit. Also notice that the regions $r < 0$ have to be excluded in order to avoid naked singularities (of the type encountered in the Schwarzschild black hole with negative mass). The black string described in [18] covers the regions $(a^-), (b^-), (c^-), (a^+)$.

Our last point concerns the expectation of the genuine AdS_3 geometry as a zero-deformation limit of the black-string metric, since the latter turns out to be a marginal deformation of AdS_3 with parameter μ. The straightforward approach consists in taking the line element in eq. (3.4a) for $\mu = 1$. It is then puzzling that the resulting extremal black-string geometry *is not* AdS_3. This apparent paradox is solved by carefully looking at the coordinate transformations that relate the black-string coordinates (r, x, t) to either the Euler coordinates (ρ, ϕ, τ) (A.5) for the $J^3\bar{J}^3$ line, or the hyperbolic coordinates (y, x, t) (A.8) for the $J^2\bar{J}^2$ line. These transformations are singular at $\mu = 1$, which therefore corresponds neither to $\kappa_3 = 1$ nor to $\kappa_2 = 1$. Put differently, $\mu = 1$ is not part of a continuous line of deformed models but marks a jump from the $J^2\bar{J}^2$ to the $J^3\bar{J}^3$ lines.

The extremal black-string solution is even more peculiar. Comparing eqs. (3.4) at $\mu = 1$ to eqs. (2.15), which describe the symmetrically null-deformed $\mathrm{SL}(2,\mathbb{R})$, we observe that the two backgrounds at hand are related by a coordinate transformation, provided $\nu = -1$.

The black string background is therefore entirely described in terms of $\mathrm{SL}(2,\mathbb{R})$ marginal symmetric deformations, and involves all three of them. The null deformation appears, however, for the extremal black string only and at a negative value of the parameter ν. The latter is the density of fundamental strings, when the deformed AdS_3 is considered within the NS5/F1 system. This might be one more sign pointing towards a possible instability in the black string [40].

Notice finally that expressions (3.4) receive $1/k$ corrections. Those have been computed in [41]. Once taken into account, they contribute in making the geometry smoother, as usual in string theory.

4. The two-parameter deformations

4.1 An interesting mix

A particular kind of asymmetric deformation is what we will call in the following *double deformation* [15, 42]. At the lagrangian level this is obtained by adding the following marginal perturbation to the WZW action:

$$\delta S = \delta\kappa^2 \int \mathrm{d}^2 z \, J\bar{J} + \mathrm{H} \int \mathrm{d}^2 z \, J\bar{I}; \tag{4.1}$$

J is a holomorphic current in the group, $\bar{J}$ is the corresponding anti-holomorphic current and $\bar{I}$ an external (to the group) anti-holomorphic current (i.e. in the right-moving heterotic sector for example). A possible way to interpret this operator consists in thinking of the double deformation as the superposition of a symmetric — or gravitational — deformation (the first addend) and of an antisymmetric one – the electromagnetic deformation. This mix is consistent because if we perform the κ deformation first, the theory keeps the $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry generated by J and $\bar{J}$ that is needed in order to allow for the $\textsc{h}$ deformation. Following this trail, we can read off the background fields corresponding to the double deformation by using at first one of the methods outlined in section 2.2 and then applying the reduction in eq. (2.17) to the resulting background fields.

The final result consists in a metric, a three-form, a dilaton and a gauge field. It is in general valid at any order in the deformation parameters κ and $\textsc{h}$ but only at leading order in α' due to the presence of the symmetric part.

Double deformations of AdS_3 where J is the time-like J^3 operator have been studied in [15]. It was there shown that the extra gravitational deformation allows to get rid of the closed time-like curves, which are otherwise present in the pure J^3 asymmetric deformation (eq. (2.19)) – the latter includes Gödel space. Here, we will focus instead on the case of double deformation generated by space-like operators, J^2 and $\bar{J}^2$.

4.2 The hyperbolic double deformation

In order to follow the above prescription for reading the background fields in the double-deformed metric let us start with the fields in eqs. (2.14). We can introduce those fields in the sigma-model action. Infinitesimal variation of the latter with respect to the parameter κ^2 enables us to reach the following expressions for the chiral currents $J^2_\kappa(z)$ and $\bar{J}^2_\kappa(\bar{z})$ at finite values of κ^2:

$$J^2_\kappa(z) = \frac{1}{\cos^2\beta + \kappa^2 \sin^2\beta}\left(\cos^2\beta\, \partial\psi - \sin^2\beta\, \partial\varphi\right), \tag{4.2}$$

$$\bar{J}^2_\kappa(\bar{z}) = \frac{1}{\cos^2\beta + \kappa^2 \sin^2\beta}\left(\cos^2\beta\, \partial\psi + \sin^2\beta\, \partial\varphi\right). \tag{4.3}$$

Note in particular that the corresponding Killing vectors (that clearly are ∂_φ and ∂_ψ) are to be rescaled as $L_2 = \frac{1}{\kappa^2}\partial_\psi - \partial_\varphi$ and $R_2 = \frac{1}{\kappa^2}\partial_\psi + \partial_\varphi$. Once the currents are known, one just has to apply the construction sketched in section 2.3 and write the background fields as follows:

$$\begin{aligned}\frac{1}{k}\,\mathrm{d}s^2 = &-\mathrm{d}\beta^2 + \cos^2\beta\frac{\left(\kappa^2 - 2\textsc{h}^2\right)\cos^2\beta + \kappa^4\sin^2\beta}{\Delta_\kappa(\beta)^2}\,\mathrm{d}\psi^2 - 4\textsc{h}^2\frac{\cos^2\beta\sin^2\beta}{\Delta_\kappa(\beta)^2}\,\mathrm{d}\psi\,\mathrm{d}\varphi + \\ &+ \sin^2\beta\frac{\cos^2\beta + \left(\kappa^2 - 2\textsc{h}^2\right)\sin^2\beta}{\Delta_\kappa(\beta)^2}\,\mathrm{d}\varphi^2\,,\end{aligned} \tag{4.4a}$$

$$\frac{1}{k}B = \frac{\kappa^2 - 2\textsc{h}^2}{\kappa^2}\frac{\cos^2\beta}{\Delta_\kappa(\beta)}\,\mathrm{d}\varphi \wedge \mathrm{d}\psi\,, \tag{4.4b}$$

$$F = 2\textsc{h}\sqrt{\frac{2k}{k_g}\frac{\sin(2\beta)}{\Delta_\kappa(\beta)^2}}\left(\kappa^2\,\mathrm{d}\psi \wedge \mathrm{d}\beta + \mathrm{d}\beta \wedge \mathrm{d}\varphi\right), \tag{4.4c}$$

$$\mathrm{e}^{2\Phi} = \frac{\sqrt{\kappa^2 - 2\textsc{h}^2}}{\Delta_\kappa(\beta)}\,, \tag{4.4d}$$

where $\Delta_\kappa(\beta) = \cos^2\beta + \kappa^2 \sin^2\beta$ as in section 2.2. In particular the dilaton, that can be obtained by imposing the one-loop beta equation is proportional to the ratio of the double deformed volume form and the AdS_3 one.

A first observation about the above background is in order here. The electric field is bounded from above since $\mathrm{H}^2 \leq \frac{\kappa^2}{2}$. As usual in string theory, tachyonic instabilities occur at large values of electric or magnetic fields, and we already observed that phenomenon in section 2.3, for purely asymmetric ($\kappa^2 = 1$) deformations. At the critical value of the parameter H, one dimension degenerates and the B-field vanishes. We are left with a two-dimensional space (with non-constant curvature) plus electric field.

The expression (4.4a) here above of the metric provides only a local description of the space-time geometry. To discuss the global structure of the whole space it is useful to perform several coordinate transformations. Firstly let us parametrize by $\kappa^2 = \lambda/(1+\lambda)$ the deformation parameter (with $\kappa < 1$ for $\lambda > 0$ and $\kappa > 1$ for $\lambda < -1$) and introduce a radial coordinate *à la* Horne and Horowitz:

$$r = \lambda + \cos^2\beta\,, \tag{4.5}$$

which obviously varies between λ and $\lambda + 1$. The expression of the metric (4.4a) becomes in terms of this new coordinate:

$$\begin{aligned}\mathrm{d}s^2 = &-\left[\left(2\mathrm{H}^2(1+\lambda)^2 - \lambda\right) + \frac{\lambda\left(\lambda - 4\mathrm{H}^2(1+\lambda)^2\right)}{r} + \frac{2\lambda^2\mathrm{H}^2(1+\lambda)^2))}{r^2}\right]\mathrm{d}\psi^2 - \\ &-(1+\lambda)\left[2\mathrm{H}^2(1+\lambda) + 1 - \frac{(1+\lambda)\left(1 + 4\mathrm{H}^2(1+\lambda)^2\right)}{r} + \frac{2(1+\lambda)^3\,\mathrm{H}^2}{r^2}\right]\mathrm{d}\varphi^2 + \\ &+ 4\mathrm{H}^2(1+\lambda)^2\left[1 - \frac{1+2\lambda}{r} + \frac{\lambda(1+\lambda)}{r^2}\right]\mathrm{d}\psi\,\mathrm{d}\varphi + \frac{1}{4(r-\lambda)(r-\lambda-1)}\mathrm{d}r^2\,. \end{aligned} \tag{4.6}$$

This expression looks close to the one discussed by Horne and Horowitz. It also represents a black string. However, it depends on more physical parameters as the expression of the scalar curvature shows:

$$\mathcal{R} = 2\frac{2r(1+2\lambda) - 7\lambda(1+\lambda) - 2\mathrm{H}^2(1+\lambda)^2}{r^2}\,. \tag{4.7}$$

This result may seem strange at first sight since, for $\kappa = 1$ and $\mathrm{H} = 0$, the metric (4.4a) is of constant Ricci (and thus scalar) curvature, corresponding to a local patch of AdS_3 while here, in the same limit, the curvature vanishes for large r. The absence of contradiction follows from the definition of the r-coordinate, becoming ill-defined for $\kappa = 1$, as it corresponds to $\lambda = \infty$.

Obviously this metric can be extended behind the initial domain of definition of the r variable. But before to discuss it, it is interesting to note that the Killing vector $\mathbf{k} = (1+\lambda)\,\partial_\psi + \lambda\,\partial_\phi \propto R_2$ is of constant square length

$$\mathbf{k}.\mathbf{k} = \lambda(1+\lambda) - 2\mathrm{H}^2(1+\lambda)^2 := \omega\,. \tag{4.8}$$

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Note that as H^2 is positive, we have the inequality $\omega < \lambda\,(1+\lambda)$. Moreover, in order to have a lorentzian signature we must impose $\omega > 0$. The fact that the Killing vector $\mathbf{k}$ is space-like and of constant length makes it a candidate to perform identifications. We shall discuss this point at the end of this section.

The constancy of the length of the Killing vector $\mathbf{k}$ suggests to make a new coordinate transformation (such that $\mathbf{k} = \partial_x$):

$$\psi = (1+\lambda)\,x + t\,, \tag{4.9a}$$

$$\varphi = t + \lambda x\,, \tag{4.9b}$$

which leads to the much simpler expression of the line element:

$$\mathrm{d}s^2 \;=\; -\frac{(r-\lambda)\,(r-\lambda-1)}{r^2}\,\mathrm{d}t^2 + \omega\left(\mathrm{d}x + \frac{1}{r}\,\mathrm{d}t\right)^2 + \frac{1}{4\,(r-\lambda)\,(r-\lambda-1)}\,\mathrm{d}r^2\,. \tag{4.10}$$

This metric is singular at $r = 0, \lambda, \lambda+1$; $r=0$ being a curvature singularity. On the other hand, the volume form is $\sqrt{\omega}/\,(2r)\;\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}r$, which indicates that the singularities at $r=\lambda$ and $r=\lambda+1$ may be merely coordinate singularities, corresponding to horizons. Indeed, it is the case. If we expand the metric, around $r=\lambda+1$, for instance, at first order (i.e. for $r=\lambda+1+\epsilon$) we obtain:

$$\mathrm{d}s^2 = \frac{\omega}{(1+\lambda)^2}(\,\mathrm{d}t + (1+\lambda)\;\mathrm{d}x)^2 - \frac{\epsilon}{(1+\lambda)^2}\,\mathrm{d}t\left[\mathrm{d}t + 2\frac{\omega}{1+\lambda}\,(\,\mathrm{d}t + (1+\lambda)\;\mathrm{d}x)\right] + \frac{1}{4\epsilon}\,\mathrm{d}r^2 \tag{4.11}$$

indicating the presence of an horizon. To eliminate the singularity in the metric, we may introduce Eddington-Finkelstein like coordinates:

$$t = (1+\lambda)\left(u \pm \frac{1}{2}\ln\epsilon\right) - \omega\xi\,, \tag{4.12a}$$

$$x = \left(1 + \frac{\omega}{1+\lambda}\right)\xi - \left(u \pm \frac{1}{2}\ln\epsilon\right). \tag{4.12b}$$

The same analysis can also be done near the horizon located at $r=\lambda$. Writing $r=\lambda+\epsilon$, the corresponding regulating coordinate transformation to use is given by:

$$t = \lambda\left(u \pm \frac{1}{2}\ln\epsilon\right) + \omega\xi\,, \tag{4.13a}$$

$$x = \left(1 - \frac{\omega}{\lambda}\right)\xi - \left(u \pm \frac{1}{2}\ln\epsilon\right). \tag{4.13b}$$

In order to reach the null Eddington-Finkelstein coordinates, we must use null rays. The geodesic equations read, in terms of a function $\Sigma^2[E,P,\varepsilon;r] = (Er-P)^2 - \left(P^2/\omega\right) - \varepsilon\,(r-\lambda)\,(r-\lambda-1)$:

$$\sigma = \int \frac{1}{4\,\Sigma[E,P,\varepsilon;r]}\,\mathrm{d}r\,, \tag{4.14a}$$

$$t = \int \frac{(Er-P)\,r}{2\,(r-\lambda)\,(r-\lambda-1)\,\Sigma[E,P,\varepsilon;r]}\,\mathrm{d}r\,, \tag{4.14b}$$

$$x = -\int \frac{(Er-P) + P/\omega}{2\,(r-\lambda)\,(r-\lambda-1)\,\Sigma[E,P,\varepsilon;r]}\,\mathrm{d}r\,, \tag{4.14c}$$

where E and P are the constant of motion associated to ∂_t and ∂_x, σ is an affine parameter and ε, equal to $1, 0, -1$, characterizes the time-like, null or space-like nature of the geodesic. Comparing these equations (with $\varepsilon = 0$ and $P = 0$) with the coordinates introduced near the horizons, we see that regular coordinates in their neighbourhoods are given by

$$t = T \pm \frac{1}{2}\left((1+\lambda)\ln|r-\lambda-1| - \lambda\ln|r-\lambda|\right), \tag{4.15a}$$

$$x = X \mp \frac{1}{2}\left(\ln|r-\lambda-1| - \ln|r-\lambda|\right), \tag{4.15b}$$

which leads to the metric

$$\mathrm{d}s^2 = \left(-1 + \frac{1+2\lambda}{r} - \frac{\lambda(1+\lambda)-\omega}{r^2}\right)\mathrm{d}T^2 + 2\frac{\omega}{r}\,\mathrm{d}X\,\mathrm{d}T + \omega\,\mathrm{d}X^2 \mp \frac{1}{r}\,\mathrm{d}T\,\mathrm{d}r\,. \tag{4.16}$$

According to the sign, we obtain incoming or outgoing null coordinates; to build a Kruskal coordinate system we have still to exponentiate them.

Obviously, we may choose the X coordinate in the metric (4.16) to be periodic without introducing closed causal curves. The question of performing more general identifications in these spaces will be discussed addressed now.

We end this section by computing the conserved charges associated to the asymptotic symmetries of our field configurations (4.4). As is well known, their expressions provide solutions of the equations of motion derived from the low-energy effective action

$$S = \int d^dx\sqrt{-g}\,\mathrm{e}^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 - \frac{k_g}{8}F^2 + \frac{\delta c}{3}\right], \tag{4.17}$$

in which we have choosen the units such that $\delta c = 12$.

Expression (4.10) for the metric is particularly appropriate to describe the asymptotic properties of the solution. In these coordinates, the various non-gravitational fields read as

$$F = \pm\frac{\sqrt{2}\mathrm{H}(1+\lambda)}{r^2\sqrt{k_g}}\,\mathrm{d}t\wedge\mathrm{d}r\,, \tag{4.18}$$

$$H = \mp\frac{\omega}{r^2}\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}r\,, \tag{4.19}$$

$$\Phi = \Phi_\star - \frac{1}{2}\ln r\,. \tag{4.20}$$

By setting $\sqrt{\omega}x = \bar{x}$ and $r = \mathrm{e}^{2\bar{\rho}}$, near infinity ($\bar{\rho} \to \infty$), the metric asymptotes the standard flat metric: $ds^2 = -dt^2 + d\bar{x}^2 + d\bar{\rho}^2$, while the fields F and H vanish and the dilaton reads $\Phi = \Phi_\star - \bar{\rho}$. This allows to interpret the asymptotic behavior of our solution (4.4) as a perturbation around the solution given by $F = 0$, $H = 0$, the flat metric and a linear dilaton: $\bar{\Phi} = \Phi_\star + f_\alpha X^\alpha$ (here $f_\alpha = (0,0,-1)$). Accordingly, we may define asymptotic charges associated to each asymptotic reductibility parameter (see [43]).

For the gauge symmetries we obtain as charges, associated to the H field

$$Q_H = \pm 2\mathrm{e}^{-2\Phi_\star}\sqrt{\omega} \tag{4.21}$$

and to the F field

$$Q_F = \pm \frac{2\sqrt{2}\mathrm{e}^{-2\Phi_\star}\textsc{h}(1+\lambda)}{\sqrt{k_g}} . \tag{4.22}$$

The first one reduces (up to normalization) for $\textsc{h} = 0$ to the result given in [18], while the second one provides an interpretation of the deformation parameter $\textsc{h}$.

Moreover, all the Killing vectors of the flat metric defining isometries that preserve the dilaton field allow to define asymptotic charges. These charges are obtained by integrating the antisymmetric tensor on the surface at infinity:

$$k_\xi^{[\mu\nu]} = \mathrm{e}^{-2\bar\Phi}\left(\xi_\sigma\, \partial_\lambda \mathcal{H}^{\sigma\lambda\mu\nu} + \frac{1}{2}\partial_\lambda \xi_\sigma\, \mathcal{H}^{\sigma\lambda\mu\nu} + 2\left(\xi^\mu h^\nu_\lambda f^\lambda - \xi^\nu h^\mu_\lambda f^\lambda\right)\right), \tag{4.23}$$

where

$$\mathcal{H}^{\sigma\lambda\mu\nu} = \bar h^{\sigma\nu}\eta^{\lambda\mu} + \bar h^{\lambda\mu}\eta^{\sigma\nu} - \bar h^{\sigma\mu}\eta^{\lambda\nu} - \bar h^{\lambda\nu}\eta^{\sigma\mu} \tag{4.24}$$

is the well known tensor sharing the symmetries of the Riemann tensor and $\bar h^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}$, while the Killing vector ξ has to fulfill the invariance condition $\xi_\alpha f^\alpha = 0$. The expression of the tensor $k_\xi^{[\mu\nu]}$ depends only on the perturbation $h_{\mu\nu}$ of the metric tensor because, on the one hand, the F and H fields appear quadratically in the lagrangian, and their background values are zero, while, on the other hand, the perturbation field for the dilaton vanishes: $\Phi = \bar\Phi$.

Restricting ourselves to constant Killing vectors, we obtain the momenta (defined for the indice $\sigma = t$ and $\bar x$)

$$P^\sigma = \int \mathrm{d}\bar x\, \mathrm{e}^{-2\bar\Phi}\left(\partial_\lambda \mathcal{H}^{\sigma\lambda t\bar\rho} - 2\eta^{\sigma t}h^\nu_{\bar\rho}\right) \tag{4.25}$$

i.e. the density of mass (μ) and momentum (ϖ) per unit length:

$$\mu = 2\mathrm{e}^{-2\Phi_\star}(1+2\lambda) \qquad \text{and} \qquad \varpi = -2\mathrm{e}^{-2\Phi_\star}\sqrt{\omega}\,. \tag{4.26}$$

Of course, if we perform identifications such that the string acquires a finite length, the momenta (4.25) become also finite.

To make an end let us notice that the expressions of μ and ϖ that we obtain differ from those given in [18] by a normalization factor but also in their dependance with respect to λ, even in the limit $\textsc{h} = 0$; indeed, the asymptotic minkowskian frames used differ from each other by a boost.

5. Discrete identifications

In the same spirit as the original BTZ construction reminded in appendix B, we would like to investigate to what extent discrete identifications could be performed in the deformed background. Necessary conditions for a solution (4.16) to remain "viable" black hole can be stated as follows:

- the identifications are to be performed along 'the orbits of some Killing vector ξ of the deformed metric
- there must be causally safe asymptotic regions (at spatial infinity)

- the norm of ξ has to be positive in some region of space-time, and chronological pathologies have to be hidden with respect to an asymptotic safe region by a horizon.

The resulting quotient space will exhibit a black hole structure if, once the regions where $\|\xi\| < 0$ have been removed, we are left with an almost geodesically complete space, the only incomplete geodesics being those ending on the locus $\|\xi\| = 0$. It is nevertheless worth emphasizing an important difference with the BTZ construction. In our situation, unlike the undeformed AdS_3 space, the initial space-time where we are to perform identifications do exhibit curvature singularities.

5.1 Discrete identifications in asymmetric deformations

Our analysis of the residual isometries in purely asymmetric deformations (section 2.3) shows that the vector ξ (eq. (B.1a)) survives only in the hyperbolic deformation, whereas ξ in eq. (B.1b) is present in the parabolic one. Put differently, non-extremal BTZ black holes allow for electric deformation, while in the extremal ones, the deformation can only be induced by an electro-magnetic wave. Elliptic deformation is not compatible with BTZ identifications.

The question that we would like to address is the following: how much of the original black hole structure survives the deformation? The answer is simple: a new chronological singularity appears in the asymptotic region of the black hole. Evaluating the norm of the Killing vector shows that a naked singularity appears. Thus the deformed black hole is no longer a viable gravitational background. Actually, whatever the Killing vector we consider to perform the identifications, we are always confronted to such pathologies.

The fate of the *asymmetric parabolic* deformation of AdS_3 is similar: there is no region at infinity free of closed time-like curves after performing the identifications.

5.2 Discrete identifications in symmetric deformations

Let us consider the *symmetric hyperbolic* deformation, whose metric is given by (4.10) with $\mathrm{H} = 0$, i.e. $\omega = \lambda\,(1+\lambda)$. This metric has two residual Killing vectors, manifestly given by ∂_t and ∂_x. We may thus, in general, consider identifications along integral lines of

$$\xi = a\,\partial_t + \partial_x\,. \tag{5.1}$$

This vector has squared norm:

$$\|\xi\|^2 = \left(\lambda\,(1+\lambda) - a^2\right) + \frac{2a\lambda\,(1+\lambda) + a^2\,(1+2\lambda)}{r}\,. \tag{5.2}$$

To be space-like at infinity the vector ξ must verify the inequality $a^2 < \lambda\,(1+\lambda)$. For definiteness, we will hereafter consider $\lambda > 0$ and $r > 0$ (the case $\lambda < -1$, $r < 0$ leads to similar conclusions, while the two other situations have to be excluded in order to avoid naked singularities, see eq. (4.10)). If $a > 0$, or $-\sqrt{\lambda\,(1+\lambda)} < a < -2\lambda\,(1+\lambda)\,/\,(1+2\lambda)$, ξ is everywhere space-like. Otherwise, it becomes time-like behind the inner horizon ($r = \lambda$), or on this horizon if $a = -\lambda$. In this situation, the quotient space will exhibit a structure similar to that of the black string, with a time-like chronological singularity (becoming light-like for $a = -\lambda$) hidden behind two horizons (or a single one for $a = -\lambda$).

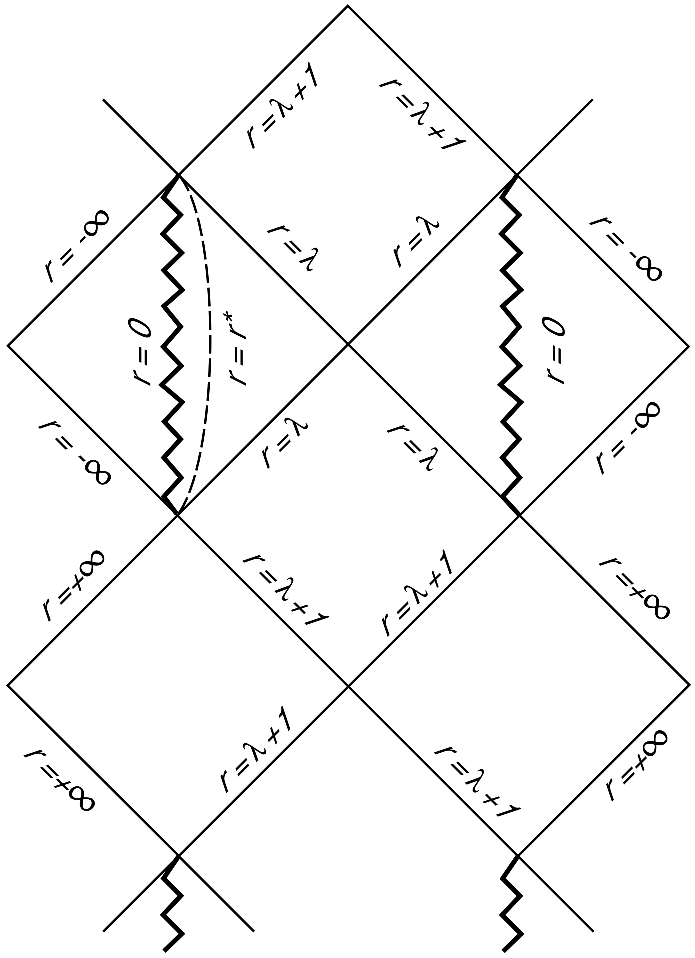

Figure 1: Penrose diagram exhibiting the global structure of the double hyperbolic deformation. The time-like curvature singularities $r = 0$ are represented, as well as the horizons, located at $r = \lambda$ and $r = \lambda + 1$. When performing identifications along orbits of Killing vectors that allow for a causally safe region at infinity, a time-like chronological singularity may appear at $r = r^*$, with $0 < r^* \leq \lambda$.

5.3 Discrete identifications in double deformations

The norm squared of the identification vector (5.1) in the metric (4.10) is

$$\|\xi\|^2 = \left(\omega - a^2\right) + 2\frac{a\omega + a^2\left(1 + 2\lambda\right)}{r} - \frac{a^2\left(\lambda\left(1 + \lambda\right) - \omega\right)}{r^2}\,. \tag{5.3}$$

Between $r = 0$ and $r = \infty$, this scalar product vanishes once and only once (if $a \neq 0$). To be space-like at infinity we have to restrict the time component of ξ to $|a| < \omega$. Near $r = 0$ it is negative, while near the inner horizon ($r = \lambda$) it takes the non-negative value $\omega\left(\lambda + a\right)^2/\lambda^2$. Accordingly, by performing identifications using this Killing vector, we will encounter a chronological singularity, located at $r = r^*$, with $0 < r^* \leq \lambda$, the singularity being of the same type as the one in the symmetric case (see figure 1).

6. Towards the exact spectra

The main guideline for exploring the black hole geometries that we have so far considered has been the presence of an underlying CFT description. This allows us to identify the background fields as the lagrangian counterparts of exact conformal field theories. In this section we will give a look to the other — algebraic — aspect of these models, showing how it is possible to write an explicit expression for the spectrum of primary operators.

Since this kind of contruction has already been carried on in [15] for the J_3 double deformation of $\mathrm{SL}(2,\mathbb{R})$, here we will focus of the J_2 deformations, giving the spectrum for the deformed theory (section (6.1)) and for a deformed theory with discrete identifications (section (6.2)). We will limit ourselves to giving the spectrum for the theory: the evaluation of the partition function, although straightforward in principle, would require the decomposition of the $\mathrm{SL}(2,\mathbb{R})$ partition function in a hyperbolic basis of characters, a still unresolved problem.

6.1 Deformed spectrum

Consider the double deformation described in section 4 for a $\mathrm{SL}(2,\mathbb{R})_k$ super-WZW model where J is the hyperbolic (space-like) J_2 current.

The evaluation of the spectrum for our deformed model is pretty straightforward once one realizes that the deformations act as $O(2,2)$ pseudo-orthogonal transformations on the charge lattice corresponding to the abelian subgroup of the $\mathfrak{sl}(2,\mathbb{R})$ heterotic model (as described in section 2). Left and right weights for the relevant lattices are (see eqs. (C.20) and (C.21)):

$$L_0 = \frac{1}{k}\left(\mu + n + \frac{a}{2}\right)^2, \tag{6.1a}$$

$$\bar{L}_0 = \frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g}\left(\bar{n} + \frac{\bar{a}}{2}\right)^2, \tag{6.1b}$$

where the anti-holomorphic part contains the contribution coming from a $\mathfrak{u}(1)$ subgroup of the heterotic gauge group.

At the lagrangian level, the infinitesimal deformation we want to describe is given by the following marginal operator:

$$\mathcal{O} = \kappa^2 \frac{\left(J^2 + \imath\psi_1\psi_3\right)}{\sqrt{k}}\frac{\bar{J}^2}{\sqrt{k+2}} + \mathrm{H}\frac{\left(J^2 + \imath\psi_1\psi_3\right)}{\sqrt{k}}\frac{\bar{I}}{\sqrt{k_g}}\,. \tag{6.2}$$

This suggests that the actual $O(2,2)$ transformation should be obtained as a boost between the holomorphic part and the result of a rotation between the two anti-holomorphic components. The deformed lattices then read:

$$L_0^{\mathrm{dd}} = \left\{\frac{1}{\sqrt{k}}\left(\mu + n + \frac{a}{2}\right)\cosh x + \left(\frac{\bar{\mu}}{\sqrt{k+2}}\cos\alpha + \frac{1}{\sqrt{k_g}}\left(\bar{n} + \frac{\bar{a}}{2}\right)\sin\alpha\right)\sinh x\right\}^2, \tag{6.3a}$$

$$\bar{L}_0^{\mathrm{dd}} = \left\{ \left(\frac{\bar\mu}{\sqrt{k+2}} \cos\alpha + \frac{1}{\sqrt{k_g}} \left(\bar n + \frac{\bar a}{2} \right) \sin\alpha \right) \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2 , \tag{6.3b}$$

where the parameters x and α can be expressed as functions of ζ and ξ as follows:

$$\begin{cases} \kappa^2 = \sinh(2x)\cos\alpha \,, \\ \mathrm{H} = \sinh(2x)\sin\alpha \,. \end{cases} \tag{6.4}$$

6.2 Twisting

The identification operation we performed in the symmetrically and double-deformed metric (as in section 5) is implemented in the string theory framework by the orbifold construction. This was already obtained in [44, 45] for the "standard" BTZ black hole that was described as a $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}$ orbifold.

In order to write the spectrum that will contain the twisted sectors, the first step consists in writing explicitly the primary fields in our theory, distinguishing between the holomorphic and anti-holomorphic parts (as it is natural to do since the construction is intrinsically heterotic).

- The holomorphic part is simply written by introducing the charge boost of eq. (6.3a) in eq. (C.16):

$$\Phi^{\mathrm{dd}}_{j\mu\nu\bar\mu\bar\nu}(z) = U_{j\mu}(z) \exp\left[\imath \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \sqrt{2}\bar Q_\alpha \sinh x \right) \vartheta_2 \right] , \tag{6.5}$$

  where $Q_\alpha = \bar\mu\sqrt{\frac{2}{k+2}}\cos\alpha + \bar\nu\sqrt{\frac{2}{k_g}}\sin\alpha$ and the dd superscript stands for double deformed

- To write the anti-holomorphic part we need at first to implement the rotation between the $\bar J^3$ and gauge current components:

$$\begin{aligned} \bar\Phi_{j\bar\mu\bar\nu}(\bar z) &= V_{j\mu}(\bar z) \mathrm{e}^{\imath\bar\mu\sqrt{2/k+2}\bar\theta_2} \mathrm{e}^{\imath\bar\nu\sqrt{2/k_g}\bar X} = \\ &= V_{j\mu}(\bar z) \mathrm{e}^{\imath\sqrt{2}\bar Q_\alpha\left(\bar\theta_2\cos\alpha + \bar X\sin\alpha\right)} \mathrm{e}^{\imath\sqrt{2}\bar Q_{\alpha-\pi/2}\left(-\bar\theta_2\sin\alpha + \bar X\cos\alpha\right)}, \end{aligned} \tag{6.6}$$

  and then realize the boost in eq. (6.3b) on the involved part:

$$\begin{aligned} \bar\Phi^{\mathrm{dd}}_{j\mu\bar\mu\nu\bar\nu}(\bar z) &= V_{j\mu} \mathrm{e}^{\imath\sqrt{2}\bar Q_{\alpha-\pi/2}\left(-\bar\theta_2\sin\alpha + \bar X\cos\alpha\right)} \times \\ &\times \exp\left[\imath \left(\sqrt{\frac{2}{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x + \sqrt{2}\bar Q_\alpha \cosh x \right) \left(\bar\theta_2\cos\alpha + \bar X\sin\alpha \right) \right]. \end{aligned} \tag{6.7}$$

Now that we have the primaries, consider the operator $W_w(z,\bar z)$ defined as follows:

$$W_w(z,\bar z) = \mathrm{e}^{-\imath\frac{k}{2}w\Delta_-\vartheta_2 + \imath\frac{k+2}{2}w\Delta_+\bar\theta_2} , \tag{6.8}$$

where $w \in \mathbb{Z}$ and $\bar{\theta}_2$ the boson corresponding to the $\bar{J}_2$ current. It is easy to show that the following OPE's hold:

$$\vartheta_2 (z) \, W_n (0, \bar{z}) \sim -\imath w \Delta_- \log z W_w (0, \bar{z}) \, , \tag{6.9}$$

$$\bar{\theta}_2 (\bar{z}) \, W_n (z, 0) \sim \imath w \Delta_+ \log \bar{z} W_w (z, 0) \, , \tag{6.10}$$

showing that $W_w (z, \bar{z})$ acts as twisting operator with winding number w (ϑ_2 and $\bar{\theta}_2$ shift by $2\pi\Delta_- w$ and $2\pi\Delta_+ w$ under $z \to e^{2\pi\imath} z$). This means that the general primary field in the $\mathrm{SL}(2,\mathbb{R})_k / \mathbb{Z}$ theory can be written as:

$$\Phi^{\mathrm{tw}}_{j\mu\bar{\mu}\nu\bar{\nu}w} (z, \bar{z}) = \Phi^{\mathrm{dd}}_{j\mu\bar{\mu}\nu\bar{\nu}} (z, \bar{z}) \, W_w (z, \bar{z}) \, . \tag{6.11}$$

where the tw superscript stands for twisted.

Having the explicit expression for the primary field, it is simple to derive the scaling dimensions which are obtained, as before, via the GKO decomposition of the Virasoro algebra $T [\mathfrak{sl}(2,\mathbb{R})] = T [\mathfrak{sl}(2,\mathbb{R}) / \mathfrak{o}(1,1)] + T [\mathfrak{o}(1,1)]$. Given that the $T [\mathfrak{sl}(2,\mathbb{R}) / \mathfrak{o}(1,1)]$ part remains invariant (and equal to $L_0 = -j(j+1)/k - \mu^2/(k+2)$ as in eq. (C.18)), the deformed weights read:

$$L_0^{\mathrm{tw}} = \left\{ \frac{k}{2\sqrt{2}} w \Delta_- + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \cosh x + \bar{Q}_\alpha \sinh x \right\}^2 , \tag{6.12a}$$

$$\begin{aligned} \bar{L}_0^{\mathrm{tw}} = & \left\{ -\frac{k+2}{2\sqrt{2}} w \Delta_+ \cos\alpha + \bar{Q}_\alpha \cosh x + \frac{1}{\sqrt{k}} \left(\mu + n + \frac{a}{2} \right) \sinh x \right\}^2 + \\ & + \left\{ \frac{k+2}{2\sqrt{2}} w \Delta_+ \sin\alpha + \bar{Q}_{\alpha - \pi/2} \right\}^2 . \end{aligned} \tag{6.12b}$$

7. Summary

The main motivation for this work has been a systematic search of black-hole structures in the moduli space of AdS_3, via marginal deformations of the $\mathrm{SL}(2,\mathbb{R})$ WZW model and discrete identifications. This allows to reach three-dimensional geometries with black-hole structure that generalize backgrounds such as the BTZ black hole [7] or the three-dimensional black string [18].

The backgrounds under consideration include a (singular) metric, a Kalb-Ramond field, a dilaton and an electric field. The latter is always bounded from above, as usual in string theory, where tachyonic instabilities are expected for large electric or magnetic fields.

We have computed parameters such as mass or charge. For backgrounds obtained by performing marginal deformations, those parameters are related to the deformation parameters. Singularities are true curvature singularities, hidden behind horizons. This is to be opposed to the BTZ black-holes, where masses and momenta are introduced by the Killing vector of the discrete identification, and where the singularity is a chronological singularity.

Discrete identifications *à la* BTZ can be superimposed to the black holes obtained by continuous deformations of AdS_3. Extra chronological singularities appear in that case,

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which force us to excise some part of the original space. This part turns out to contain the locus of the curvature singularity. It is worth stressing that for certain range of the deformation parameters, *naked* singularities appear.

Although the geometrical view point has been predominating, the guideline for our study comes from the underlying CFT structure. This has enabled us to provide both a geometrical and an algebraical description in terms of the spectrum of the string primaries.

Since we are dealing with the extension of AdS_3 one may wonder about a possible holographic interpretation for the exact string backgrounds at hand, aiming at generalizing the usual AdS/CFT correspondence. A major obstruction to this is due to the asymptotic flatness of the geometries. Hence, it is not clear how to find a suitable boundary map.

Interesting questions that we did not address, which are in principle within reach, are those dealing with the thermodynamical properties of the above black holes, for which a microscopic interpretation in terms of string states should be tractable.

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A. AdS_3 coordinate patches

A.1 AdS_3 $\mathrm{SL}(2,\mathbb{R})$

The commutation relations for the generators of the $\mathrm{SL}(2,\mathbb{R})$ algebra are

$$\left[J^1, J^2\right] = -\imath J^3 \qquad \left[J^2, J^3\right] = \imath J^1 \qquad \left[J^3, J^1\right] = \imath J^2 \,. \tag{A.1}$$

The three-dimensional anti-de Sitter space is the universal covering of the $\mathrm{SL}(2,\mathbb{R})$ group manifold. The latter can be embedded in a lorentzian flat space with signature $(-,+,+,-)$ and coordinates (x^0, x^1, x^2, x^3):

$$g = L^{-1} \begin{pmatrix} x^0 + x^2 & x^1 + x^3 \\ x^1 - x^3 & x^0 - x^2 \end{pmatrix}, \tag{A.2}$$

where L is the radius of AdS_3.

The isometry group of the $\mathrm{SL}(2,\mathbb{R})$ group manifold is generated by left or right actions on g: $g \to hg$ or $g \to gh$ $\forall h \in \mathrm{SL}(2,\mathbb{R})$. From the four-dimensional point of view, it is generated by the Lorentz boosts or rotations $\zeta_{ab} = i\left(x_a \partial_b - x_b \partial_a\right)$ with $x_a = \eta_{ab} x^b$. We list here explicitly the six Killing vectors, as well as the group action they correspond to:

$$L_1 = \frac{\imath}{2}\left(\zeta_{32} - \zeta_{01}\right), \qquad g \to \mathrm{e}^{-\frac{\lambda}{2}\sigma^1} g, \tag{A.3a}$$

$$L_2 = \frac{\imath}{2}\left(-\zeta_{31} - \zeta_{02}\right), \qquad g \to \mathrm{e}^{-\frac{\lambda}{2}\sigma^3} g, \tag{A.3b}$$

$$L_3 = \frac{\imath}{2}\left(\zeta_{03} - \zeta_{12}\right), \qquad g \to \mathrm{e}^{\imath\frac{\lambda}{2}\sigma^2} g, \tag{A.3c}$$

$$R_1 = \frac{\imath}{2}\left(\zeta_{01} + \zeta_{32}\right), \qquad g \to g\mathrm{e}^{\frac{\lambda}{2}\sigma^1}, \tag{A.3d}$$

$$R_2 = \frac{\imath}{2}\left(\zeta_{31} - \zeta_{02}\right), \qquad g \to g\mathrm{e}^{-\frac{\lambda}{2}\sigma^3}, \tag{A.3e}$$

$$R_3 = \frac{\imath}{2}\left(\zeta_{03} + \zeta_{12}\right), \qquad g \to g\mathrm{e}^{\imath\frac{\lambda}{2}\sigma^2}. \tag{A.3f}$$

Both sets satisfy the algebra (A.1) (once multiplied by $-\imath$). The norms of the Killing vectors are the following:

$$\|L_1\|^2 = \|R_1\|^2 = \|L_2\|^2 = \|R_2\|^2 = -\|L_3\|^2 = -\|R_3\|^2 = \frac{L^2}{4}. \tag{A.4}$$

Moreover $L_i \cdot L_j = 0$ for $i \neq j$ and similarly for the right set. Left vectors are not orthogonal to right ones.

The isometries of the $\mathrm{SL}(2,\mathbb{R})$ group manifold turn into symmetries of the $\mathrm{SL}(2,\mathbb{R})_k$ WZW model, where they are realized in terms of conserved currents.[10] The reader will find details on those issues in the appendices of [16].

A.2 "Symmetric" coordinates

One introduces Euler-like angles by

$$g = \mathrm{e}^{\imath\frac{\tau+\phi}{2}\sigma^2}\mathrm{e}^{\rho\sigma^1}\mathrm{e}^{\imath\frac{\tau-\phi}{2}\sigma^2}, \tag{A.5}$$

which provide good global coordinates for AdS_3 when $\tau \in]-\infty, +\infty[$, $\rho \in [0, \infty[$, and $\phi \in [0, 2\pi]$. In Euler angles, the invariant metric reads:

$$\mathrm{d}s^2 = L^2\left[-\cosh^2\rho\, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\phi^2\right]. \tag{A.6}$$

The Ricci scalar of the corresponding Levi-Civita connection is $\mathcal{R} = -6/L^2$. The volume form reads:

$$\omega_{[3]} = \frac{L^3}{2}\sinh 2\rho\, \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\tau, \tag{A.7}$$

whereas $L_3 = \frac{1}{2}\left(\partial_\tau + \partial_\phi\right)$ and $R_3 = \frac{1}{2}\left(\partial_\tau - \partial_\phi\right)$.

Another useful, although not global, set of coordinates is defined by

$$g = \mathrm{e}^{\frac{\psi-\varphi}{2}\sigma^3}\mathrm{e}^{\imath\beta\sigma^1}\mathrm{e}^{\frac{\psi+\varphi}{2}\sigma^3}, \tag{A.8}$$

[10] When writing actions a choice of gauge for the NS potential is implicitly made, which breaks part of the symmetry: boundary terms appear in the transformations. These must be properly taken into account in order to reach the conserved currents. Although the expressions for the latter are not unique, they can be put in an improved-Noether form, in which they have only holomorphic (for L_i's) or anti-holomorphic (for R_j's) components.

(ψ and φ *are not* compact coordinates). The metric reads:

$$\mathrm{d}s^2 = L^2 \left[\cos^2\beta\, \mathrm{d}\psi^2 - \mathrm{d}\beta^2 + \sin^2\beta\, \mathrm{d}\varphi^2\right], \tag{A.9}$$

with volume form

$$\omega_{[3]} = \frac{L^3}{2}\sin 2\beta\, \mathrm{d}\beta \wedge \mathrm{d}\psi \wedge \mathrm{d}\varphi\,. \tag{A.10}$$

Now $L_2 = \frac{1}{2}\left(\partial_\psi - \partial_\varphi\right)$ and $R_2 = \frac{1}{2}\left(\partial_\psi + \partial_\varphi\right)$.

Finally, the Poincaré coordinate system is defined by

$$\begin{cases} x^0 + x^2 = \dfrac{L}{u}\,, \\ x^0 - x^2 = Lu + \dfrac{Lx^+x^-}{u}\,, \\ x^1 \pm x^3 = \dfrac{Lx^\pm}{u}\,. \end{cases} \tag{A.11}$$

For $\{u, x^+, x^-\} \in \mathbb{R}^3$, the Poincaré coordinates cover once the $\mathrm{SL}(2\mathbb{R})$ group manifold. Its universal covering, AdS_3, requires an infinite number of such patches. Moreover, these coordinates exhibit a Rindler horizon at $|u| \to \infty$; the conformal boundary is at $|u| \to 0$. Now the metric reads:

$$\mathrm{d}s^2 = \frac{L^2}{u^2}\left(\mathrm{d}u^2 + \mathrm{d}x^+\,\mathrm{d}x^-\right), \tag{A.12}$$

and the volume form:

$$\omega_{[3]} = \frac{L^3}{2u^3}\,\mathrm{d}u \wedge \mathrm{d}x^+ \wedge \mathrm{d}x^-\,. \tag{A.13}$$

We also have $L_1 + L_3 = -\,\partial_-$ and $R_1 + R_3 = \partial_+$.

A.3 "Asymmetric" coordinates

The above three sets of AdS_3 coordinates are suitable for implementing symmetric parabolic, elliptic or hyperbolic deformations, respectively driven by $\left(J^1 + J^3\right)\left(\bar{J}^1 + \bar{J}^3\right)$, $J^3\bar{J}^3$ or $J^2\bar{J}^2$. For asymmetric elliptic or hyperbolic deformations, we must use different coordinate systems, where the structure of AdS_3 as a Hopf fibration is more transparent. They are explicitly described in the following.

- The coordinate system used to describe the elliptic asymmetric deformation is defined as follows:

$$\begin{cases} \dfrac{x_0}{L} = \cosh\dfrac{\rho}{2}\cosh\dfrac{\phi}{2}\cos\dfrac{t}{2} - \sinh\dfrac{\rho}{2}\sinh\dfrac{\phi}{2}\sin\dfrac{t}{2}\,, \\ \dfrac{x_1}{L} = -\sinh\dfrac{\rho}{2}\sinh\dfrac{\phi}{2}\cos\dfrac{t}{2} - \cosh\dfrac{\rho}{2}\sinh\dfrac{\phi}{2}\sin\dfrac{t}{2}\,, \\ \dfrac{x_2}{L} = -\cosh\dfrac{\rho}{2}\sinh\dfrac{\phi}{2}\cos\dfrac{t}{2} + \sinh\dfrac{\rho}{2}\cosh\dfrac{\phi}{2}\sin\dfrac{t}{2}\,, \\ \dfrac{x_3}{L} = -\sinh\dfrac{\rho}{2}\sinh\dfrac{\phi}{2}\cos\dfrac{t}{2} - \cosh\dfrac{\rho}{2}\cosh\dfrac{\phi}{2}\sin\dfrac{t}{2}\,. \end{cases} \tag{A.14}$$

  The metric now reads:

$$\mathrm{d}s^2 = \frac{L^2}{4}\left(\mathrm{d}\rho^2 + \mathrm{d}\phi^2 - \mathrm{d}t^2 - 2\sinh\rho\,\mathrm{d}t\,\mathrm{d}\phi\right), \tag{A.15}$$

and the corresponding volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh\rho \, \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}t. \tag{A.16}$$

This coordinate system is such that the t-coordinate lines coincide with the integral curves of the Killing vector $L_3 = -\partial_t$, whereas the ϕ-lines are the curves of $R_2 = \partial_\phi$.

- The coordinate system used to describe the asymmetric hyperbolic deformation is defined as follows:

$$\begin{cases} \dfrac{x_0}{L} = \cosh\dfrac{r}{2}\cosh\dfrac{x}{2}\cos\dfrac{\tau}{2} + \sinh\dfrac{r}{2}\sinh\dfrac{x}{2}\sin\dfrac{\tau}{2}\,, \\ \dfrac{x_1}{L} = -\sinh\dfrac{r}{2}\cosh\dfrac{x}{2}\cos\dfrac{\tau}{2} + \cosh\dfrac{r}{2}\sinh\dfrac{x}{2}\sin\dfrac{\tau}{2}\,, \\ \dfrac{x_2}{L} = -\cosh\dfrac{r}{2}\sinh\dfrac{x}{2}\cos\dfrac{\tau}{2} - \sinh\dfrac{r}{2}\cosh\dfrac{x}{2}\sin\dfrac{\tau}{2}\,, \\ \dfrac{x_3}{L} = \sinh\dfrac{r}{2}\sinh\dfrac{x}{2}\cos\dfrac{\tau}{2} - \cosh\dfrac{r}{2}\cosh\dfrac{x}{2}\sin\dfrac{\tau}{2}\,. \end{cases} \tag{A.17}$$

For $\{r, x, \tau\} \in \mathbb{R}^3$, this patch covers exactly once the whole AdS_3, and is regular everywhere [46]. The metric is then given by

$$\mathrm{d}s^2 = \frac{L^2}{4}\left(\mathrm{d}r^2 + \mathrm{d}x^2 - \mathrm{d}\tau^2 + 2\sinh r \, \mathrm{d}x \, \mathrm{d}\tau\right), \tag{A.18}$$

and correspondingly the volume form is

$$\omega_{[3]} = \frac{L^3}{8} \cosh r \, \mathrm{d}r \wedge \mathrm{d}x \wedge \mathrm{d}\tau. \tag{A.19}$$

In this case the x-coordinate lines coincide with the integral curves of the Killing vector $L_2 = \partial_x$, whereas the τ-lines are the curves of $R_3 = -\partial_\tau$.

B. The BTZ black hole

In the presence of isometries, discrete identifications provide alternatives for creating new backgrounds. Those have the *same* local geometry, but differ with respect to their global properties. Whether these identifications can be implemented as orbifolds at the level of the underlying two-dimensional string model is very much dependent on each specific case.

For AdS_3, the most celebrated geometry obtained by discrete identification is certainly the BTZ black hole [7]. The discrete identifications are made along the integral lines of the following Killing vectors (see eqs. (A.3)):

$$\text{non-extremal case}: \quad \xi = (r_+ + r_-)\, R_2 - (r_+ - r_-)\, L_2\,, \tag{B.1a}$$

$$\text{extremal case}: \quad \xi = 2r_+ R_2 - (R_1 - R_3) - (L_1 + L_3)\,. \tag{B.1b}$$

In the original BTZ coordinates, the metric reads:

$$\mathrm{d}s^2 = L^2\left[-f^2(r)\,\mathrm{d}t^2 + f^{-2}(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\varphi - \frac{r_+ r_-}{r^2}\,\mathrm{d}t\right)^2\right], \tag{B.2}$$

with

$$f(r) = \frac{1}{r}\sqrt{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}\,. \tag{B.3}$$

In this coordinate system,

$$\partial_\varphi \equiv \xi\,, \qquad \partial_t \equiv -\left(r_+ + r_-\right) R_2 - \left(r_+ - r_-\right) L_2 \qquad \text{and} \qquad r^2 \equiv \|\xi\|\;. \tag{B.4}$$

In AdS_3 φ is not a compact coordinate. The discrete identification makes φ an angular variable, $\varphi \cong \varphi + 2\pi$, which imposes to remove the region with $r^2 < 0$. The BTZ geometry describes a three-dimensional black hole, with mass M and angular momentum J, in a space-time that is locally (and asymptotically) anti-de Sitter. The chronological singularity at $r = 0$ is hidden behind an inner horizon at $r = r_-$, and an outer horizon at $r = r_+$. Between these two horizons, r is time-like. The coordinate t becomes space-like inside the ergosphere, when $r^2 < r_{\text{erg}}^2 \equiv r_+^2 + r_-^2$. The relation between M, J and $r_\pm$ is as follows:

$$r_\pm^2 = \frac{ML}{2}\left[1 \pm \sqrt{1 - \left(\frac{J}{ML}\right)^2}\,\right]. \tag{B.5}$$

Extremal black holes have $|J| = ML$ ($r_+ = r_-$). In the special case $J = ML = 0$ one finds the near-horizon geometry of the five-dimensional NS5/F1 stringy black hole in its ground state. Global AdS_3 is obtained for $J = 0$ and $ML = -1$.

Many subtleties arise, which concern e.g. the appearance of closed time-like curves in the excised region of negative r^2 (where ∂_φ would have been time-like) or the geodesic completion of the manifold; a comprehensive analysis of these issues can be found in [8]. At the string-theory level, the BTZ identification is realized as an *orbifold* projection, which amounts to keeping invariant states and adding twisted sectors [44, 45].

Besides the BTZ solution, other locally AdS_3 geometries are obtained, by imposing identification under purely left (or right) isometries, refereed to as self-dual (or anti-self-dual) metrics. These were studied in [46]. Their classification and isometries are exactly those of the asymmetric deformations studied in the present chapter. The Killing vector used for the identification is (A) time-like (elliptic), (B) space-like (hyperbolic) or (C) null (parabolic), and the isometry group is $U(1) \times SL(2,\mathbb{R})$. It was pointed out in [46] that the resulting geometry was free of closed time-like curves only in the case (B).

C. Spectrum of the $SL(2,\mathbb{R})$ super-wzw model

In this appendix we give a reminder of the superconformal WZW model on $SL(2,\mathbb{R})_k$ (for a recent discussion see [47]). The affine extension of the $\mathfrak{sl}(2,\mathbb{R})$ algebra at level k is obtained by considering two sets of holomorphic and anti-holomorphic currents of dimension one, defined as

$$J^{\mathrm{M}}(z) = k\left\langle T^{\mathrm{M}}, \mathrm{Ad}_g g^{-1}\partial g\right\rangle\,, \qquad \bar{J}^{\mathrm{M}}(\bar{z}) = k\left\langle T^{\mathrm{M}}, g^{-1}\bar{\partial} g\right\rangle\,, \tag{C.1}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product (Killing form) in $\mathfrak{sl}(2,\mathbb{R})$, $\{T^{\mathrm{M}}\}$ is a set of generators of the algebra that for concreteness we can choose as follows:

$$T^1 = \sigma^1\,, \qquad T^2 = \sigma^3\,, \qquad T^3 = \sigma^2\,. \tag{C.2}$$

Each set satisfies the OPE

$$J^{\mathrm{M}}(z)\, J^{\mathrm{N}}(w) \sim \frac{k\delta^{\mathrm{MN}}}{2\,(z-w)^2} + \frac{f^{\mathrm{MN}}{}_{\mathrm{P}} J^{\mathrm{P}}(w)}{z-w}\,, \tag{C.3}$$

where $f^{\mathrm{MN}}{}_{\mathrm{P}}$ are the structure constants of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. The chiral algebra contains the Virasoro operator (stress tensor) obtained by the usual Sugawara construction:

$$T(z) = \sum_{\mathrm{M}} \frac{:J^{\mathrm{M}}J^{\mathrm{M}}:}{k-2}\,. \tag{C.4}$$

A heterotic model is built if we consider a left-moving $\mathcal{N}=1$ extension, obtained by adding 3 free fermions which transform in the adjoint representation. More explicitly:

$$T(z) = \sum_{\mathrm{M}} \frac{:J^{\mathrm{M}}J^{\mathrm{M}}:}{k-2} + :\psi_{\mathrm{M}}\partial\psi_{\mathrm{M}}:\,, \tag{C.5}$$

$$G(z) = \frac{2}{k}\left(\sum_{\mathrm{M}} J^{\mathrm{M}}\psi_{\mathrm{M}} - \frac{\imath}{3k}\sum_{\mathrm{MNP}} f^{\mathrm{MNP}} :\psi_{\mathrm{M}}\psi_{\mathrm{N}}\psi_{\mathrm{P}}:\right). \tag{C.6}$$

On the right side, instead of superpartners, we add a right-moving current with total central charge $c=16$.

Let us focus on the left-moving part. The supercurrents are given by $\psi_{\mathrm{M}} + \theta\sqrt{2/k}\mathcal{J}_{\mathrm{M}}$ where:

$$\mathcal{J}_{\mathrm{M}} = J^{\mathrm{M}} - \frac{\imath}{2}\sum_{\mathrm{NP}} \epsilon^{\mathrm{MNP}}\psi_{\mathrm{N}}\psi_{\mathrm{P}}\,; \tag{C.7}$$

it should be noted that the bosonic J^{M} currents generate an affine $\mathfrak{sl}(2,\mathbb{R})$ algebra at level $k+2$, while the level for the total $\mathcal{J}_{\mathrm{M}}$ currents is k.

Let us now single out the operator that we used for both the deformation (eqs. (4.4a)) and the identifications (section 5.3):

$$\mathcal{J}_2 = J^2 + \imath\psi_1\psi_3\,. \tag{C.8}$$

Let us now bosonize these currents as follows:

$$\mathcal{J}_2 = -\sqrt{\frac{k}{2}}\partial\vartheta_2\,, \tag{C.9}$$

$$J^2 = -\sqrt{\frac{k+2}{2}}\partial\theta_2\,, \tag{C.10}$$

$$\psi_1\psi_3 = \partial H, \tag{C.11}$$

and introduce a fourth free boson X so to separate the ϑ_2 components both in θ_2 and H:

$$\imath H = \sqrt{\frac{2}{k}}\vartheta_2 + \imath\sqrt{\frac{k+2}{k}}X, \tag{C.12}$$

$$\theta_2 = \sqrt{\frac{2}{k}}\left(\sqrt{\frac{k+2}{2}}\vartheta_2 + \imath X\right). \tag{C.13}$$

A primary field $\Phi_{j\mu\bar{\mu}}$ of the bosonic $\mathrm{SL}(2,\mathbb{R})_{k+2}$ with eigenvalue μ with respect to J^2 and $\bar{\mu}$ with respect to $\bar{J}^2$ obeys by definition

$$J^2(z)\,\Phi_{j\mu\bar{\mu}}(w,\bar{w}) \sim \frac{\mu\Phi_{j\mu\bar{\mu}}(w,\bar{w})}{z-w}\,, \tag{C.14a}$$

$$\bar{J}^2(\bar{z})\,\Phi_{j\mu\bar{\mu}}(w,\bar{w}) \sim \frac{\bar{\mu}\Phi_{j\mu\bar{\mu}}(w,\bar{w})}{\bar{z}-\bar{w}}. \tag{C.14b}$$

Since $\Phi_{j\mu\bar{\mu}}$ is purely bosonic, the same relation holds for the supercurrent:

$$\mathcal{J}_2(z)\,\Phi_{j\mu\bar{\mu}}(w,\bar{w}) \sim \frac{\mu\Phi_{j\mu\bar{\mu}}(w,\bar{w})}{z-w}\,. \tag{C.15}$$

Consider now the holomorphic part of $\Phi_{j\mu\bar{\mu}}(z,\bar{z})$. If $\Phi_{j\mu}$ is viewed as a primary in the SWZW model, we can use the parafermion decomposition as follows:

$$\Phi_{j\mu}(z) = U_{j\mu}(z)\,\mathrm{e}^{\imath\mu\sqrt{2/k}\vartheta_2}\,, \tag{C.16}$$

where $U_{j\mu}(z)$ is a primary of the superconformal $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$. On the other hand, we can just consider the bosonic WZW and write:

$$\Phi_{j\mu}(z) = V_{j\mu}(z)\,\mathrm{e}^{\imath\mu\sqrt{2/(k+2)}\theta_2} = V_{j\mu}(z)\,\mathrm{e}^{\imath\frac{2m}{k+2}\sqrt{\frac{k+2}{k}}X+\imath\mu\sqrt{2/k}\vartheta_2}\,, \tag{C.17}$$

where now $V_{j\mu}(z)$ is a primary of the bosonic $\mathrm{SL}(2,\mathbb{R})_{k+2}/\mathrm{U}(1)$. The scaling dimension for this latter operator (i.e. its eigenvalue with respect to L_0) is then given by:

$$\Delta(V_{j\mu}) = -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2}\,. \tag{C.18}$$

An operator in the full supersymmetric $\mathrm{SL}(2,\mathbb{R})_k$ theory is then obtained by adding the $\psi^1\psi^3$ fermionic superpartner contribution:

$$\Phi_{j\mu\nu}(z) = \Phi_{j\mu}(z)\,\mathrm{e}^{\imath\nu H} = V_{j\mu}(z)\,\mathrm{e}^{\imath\left(\frac{2\mu}{k+2}+\nu\right)\sqrt{\frac{k+2}{k}}X}\,\mathrm{e}^{\imath\sqrt{2/k}(\mu+\nu)\vartheta_2} \tag{C.19}$$

that is an eigenvector of $\mathcal{J}_2$ with eigenvalue $\mu+\nu$ where $\mu\in\mathbb{R}$ and ν can be decomposed as $\nu = n + a/2$ with $n\in\mathbb{N}$ and $a\in\mathbb{Z}_2$ depending on whether we consider the NS or R sector. The resulting spectrum can be read directly as:

$$\begin{aligned}\Delta(\Phi_{j\mu n}(z)) &= -\frac{j(j+1)}{k} - \frac{\mu^2}{k+2} - \frac{k+2}{2k}\left(\frac{2\mu}{k+2}+n+\frac{a}{2}\right)^2 + \frac{1}{k}\left(\mu+n+\frac{a}{2}\right)^2\\ &= -\frac{j(j+1)}{k} - \frac{1}{2}\left(n+\frac{a}{2}\right)^2.\end{aligned} \tag{C.20}$$

Of course the last expression was to be expected since it is the sum of the $\mathfrak{sl}(2,\mathbb{R})_{k+2}$ Casimir and the contribution of a light-cone fermion. Nevertheless the preceding construcion is useful since it allowed us to isolate the $\mathcal{J}_2$ contribution to the spectrum $(\mu+\nu)^2/k$.

The right-moving part of the spectrum is somewhat simpler since there are no superpartners. This means that we can repeat our construction above and the eigenvalue of the $\bar{L}_0$ operator is simply obtained by adding to the dimension in eq. (C.18) the contribution of the $\bar{J}^2$ operator and of some U(1) coming from the gauge sector:

$$\bar{\Delta}\left(\bar{\Phi}_{j\bar{\mu}\bar{n}}(\bar{z})\right) = -\frac{j(j+1)}{k} - \frac{\bar{\mu}^2}{k+2} + \left\{\frac{\bar{\mu}^2}{k+2} + \frac{1}{k_g}\left(\bar{n}+\frac{\bar{a}}{2}\right)^2\right\}, \tag{C.21}$$

where again $\bar{n}\in\mathbb{N}$ and $\bar{a}\in\mathbb{Z}_2$ depending on the sector.

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